UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3801**

ASSESSMENT PATTERN : **MATH3801B**

MODULE NAME : **Logic**

DATE : **09-May-14**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2013/14-MATH3801B-001-EXAM-277

TURN OVER

All questions may be attempted, but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define $\mathcal{L}$, the set of formulae in the general first order predicate language.

   (b) For each of the following strings, use the definition of $\mathcal{L}$ in order to determine whether or not the string is a formula:
   (below, x, y, z are variable symbols and P is a unary predicate symbol)

   (i) $\forall x \Rightarrow Px \Rightarrow \neg PyPz$

   (ii) $\forall x \Rightarrow Px\forall yPy\neg Pz$

   (c) Convert each of the following formulae of $\mathcal{L}_{math}$ to a formula in $\mathcal{L}$:
   (below, x, y, z are variable symbols and P is a unary predicate symbol)

   (i) $((Px \vee Py) \vee Pz)$

   (ii) $(\exists y)\,((Px \Rightarrow Py) \wedge Py)$

   (d) Define the weight of a string.

   (e) Show that any formula has weight -1.

2. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.

   (b) Suppose that S is a subset of the set $\mathcal{L}_0$ of propositions and that α is a proposition. State what it means to say that the semantic implication $S \models \alpha$ holds.

   (c) Use the semantic tableaux method to determine whether or not each of the following semantic implications holds (where α, β, γ are distinct primitive propositions). If a semantic implication does not hold, describe a (type of) valuation for which it fails to hold.

   (i) $\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\} \models (\beta \Rightarrow (\alpha \Rightarrow \gamma))$

   (ii) $\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\} \models (\gamma \Rightarrow (\beta \Rightarrow \alpha))$

   (iii) $\{(\neg\alpha) \wedge (\beta \Rightarrow \gamma)\,,\, (\neg\beta) \Rightarrow \alpha\} \models (\beta \wedge \gamma)$

3. (a) Give a direct proof of the following (for propositions α, β):

$$\{\alpha \Rightarrow (\alpha \Rightarrow (\neg\beta))\} \vdash (\alpha \Rightarrow (\neg\beta))$$

(You may assume the validity of the following theorem: $\vdash (\alpha \Rightarrow \alpha)$.)

(b) (i) State the Deduction Theorem for propositional logic.

(ii) Use the Deduction Theorem to show that (for propositions α, β, γ, δ):

$$\{\alpha \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow \delta)\} \vdash \alpha \Rightarrow (\gamma \Rightarrow \delta)$$

(c) (i) State what it means for a set S of propositions to be consistent.

(ii) State and prove the Adequacy Theorem for propositional logic. (You may assume that any consistent set of propositions has a model.)

4. (a) State and prove the Compactness Theorem for first order predicate logic. (You may assume the following form of the Completeness Theorem for first order predicate logic: If S is a set of sentences in a first order predicate language, then S is consistent if and only if S has a model.)

(b) Describe a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure U is a (normal) model of the theory if and only if U is a poset (with equality) containing at most 5 elements.

(c) Does there exist a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure U is a (normal) model of the theory if and only if U is a poset (with equality) containing finitely many elements? Explain your answer; you may quote, without proof, any relevant results.

5. (a) Define the notion of a register machine, giving also a description of what a program is and of the types of instructions associated to the states of a program.

(b) Describe a way that may be used to uniquely encode each program, by a natural number.

(c) Consider the following function:

$$f : \mathbb{N}_0 \to \mathbb{N}_0, \quad f(m) = 4$$

(i) Describe a register machine that shows that f is computable.

(ii) Find a natural number that uniquely encodes the described program.

(d) Consider the following function:

$$g : \mathbb{N}_0 \to \mathbb{N}_0, \quad g(m) = 3m + 2$$

Describe a register machine that shows that g is computable.